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LETTER TO THE EDITOR

Singular sector of the Burgers–Hopf hierarchy and deformations of hyperelliptic curves

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Abstract

We discuss the structure of shock singularities of the Burgers–Hopf hierarchy. It is shown that the set of singular solutions defines a stratification of the affine space of the flow parameters in the hierarchy. The stratification is associated with the Birkhoff decomposition of the Grassmannian given by the set of linear spaces spanned by the hierarchy. We then construct integrable hierarchy on each stratum and demonstrate that it describes a deformation of a hyperelliptic curve parametrizing the stratum. The hierarchy is called the hidden Burgers–Hopf hierarchy, and we found the Riemann invariant form and the hodograph solution.

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1. Introduction: singular solutions of the BH hierarchy

The Burgers–Hopf (BH) hierarchy defined by

$$\frac{\partial u}{\partial t_n} = c_n u^n \frac{\partial u}{\partial x} \quad \text{with} \quad c_n = (-1)^n \frac{(2n+1)!!}{2^n n!} \quad n = 0, 1, 2, \dots$$

plays an important role in the study of a wide range of phenomena in physics from hydrodynamics to topological field theory (for examples, see [7, 8, 23]). It is clear that the BH hierarchy can be obtained by the dispersionless limit of the KdV hierarchy as well as the dissipationless limit of the Burgers hierarchy. However, the present study does not require any information of the original hierarchy.

Using the method of characteristic, the solution of the hierarchy can be expressed by the hodograph form with $t_0 \equiv x$,

$$\Omega(u, x, t_1, t_2, \dots) := x + \sum_{j=1}^{\infty} c_j u^j t_j - f(u) = 0 \quad (1.1)$$

where $f(u)$ may be determined by the initial data, e.g. $u(x, 0, 0, \dots) = f^{-1}(x)$ for monotone data. Then the BH hierarchy gives a deformation of the data $x = f(u)$ in each t_j -direction, and this defines an infinite-dimensional surface given by (1.1).

Here we are interested in the class of solutions of the BH hierarchy with singularity of shock wave type. The singularity of this type corresponds to a point on the surface where the derivative $\partial u/\partial x$ blows up at this point, that is, equation (1.1) is not invertible for u . This condition is equivalent to $\Omega' := \partial\Omega/\partial u = 0$ at the point, which implies that the solution u can be expressed as a function of (t_1, t_2, \dots) . Thus the conditions $\Omega = 0$ and $\Omega' = 0$ define a singular sector of codimension ≥ 1 in \mathbb{C}^∞ . The affine space \mathbb{C}^∞ is then stratified with those singular sectors as follows. Let us first define the solution sets,

$$U_n := \{u \mid \Omega^{(j)} = 0 \ (0 \leq j \leq n), \Omega^{(n+1)} \neq 0\}$$

where $\Omega^{(i)} = \partial^i \Omega / \partial u^i$. Then the singular sector of codimension m can be defined as

$$Z_m := \left\{ (t_0, t_1, t_2, \dots) \in \mathbb{C}^\infty \mid u \in U_j \text{ such that } m = \sum_{j=1}^\infty j|U_j| \right\}. \tag{1.2}$$

where $|U_j|$ is the number of solutions in U_j . For example, Z_0 consists of the points where $\Omega(u) = 0$ has only simple roots, and the points in Z_1 correspond to the case where $\Omega(u) = 0$ and $\Omega'(u) = 0$ has only one common root and no roots for $\Omega^{(j)}(u) = 0, j \geq 2$. The closure of the set of those singular sectors then forms a decomposition of the affine space \mathbb{C}^∞ , i.e.

$$\mathbb{C}^\infty = \overline{\bigcup_{m \geq 0} Z_m} \text{ disjoint union.}$$

In order to show some explicit structure of the set Z_m , let us consider the case where $f(u)$ is given by an $(N + 1)$ th degree of polynomial in u . Then on an $(N + 1)$ -dimensional subspace of \mathbb{C}^∞ with $t_n = 0$ for $n \geq N + 1$, equation (1.1) can be written as

$$\Omega_{N+1}(u, x, t_1, \dots, t_N) = x + \sum_{j=1}^N c_j u^j t_j = u^{N+1} + \sum_{j=0}^N b_j u^j. \tag{1.3}$$

Since all the coefficients b_j on the right-hand side can be absorbed by shifting the times, $(x - b_0, t_1 - (b_1/c_1), t_2 - (b_2/c_2), \dots)$, we just set all $b_j = 0$. Then we consider the singular sectors, denoted as Z_m^{N+1} , on the subspace \mathbb{C}^{N+1} . Equation (1.3) is well known as a universal unfolding of a singularity of A_N -type [5]. The singularity of this type includes cusp, swallow tail and butterfly for $N = 2, 3$ and 4. As an example, we just consider the case with $N = 2$.

Example 1.1. The hodograph solution (1.3) with $N = 2$ is given by

$$\Omega_3(u, x, t_1, t_2) = x - \frac{3}{2}ut_1 + \frac{15}{8}u^2t_2 - u^3 = 0.$$

Then the singular sector Z_1^3 is given by

$$Z_1^3 = Z_1^{3+} \cup Z_1^{3-} \quad \text{with} \quad Z_1^{3\pm} = \{(x, t_1, t_2) \mid \Omega_3(u^\pm, x, t_1, t_2) = 0, u^+ \neq u^-\}$$

where u^\pm are the roots of $\Omega'_3 = 0$, i.e.

$$u^\pm = \frac{1}{8} \left(5t_2 \pm \sqrt{25t_2^2 - 32t_1} \right).$$

The sets $Z_1^{3\pm}$ obviously intersect on the double points of $u^+ = u^-$, and form a cusp singularity. The intersection provides the higher singular sector Z_2^3 which is given by a twisted cubic curve:

$$Z_2^3 = \{(x, t_1, t_2) \mid x = u^3, t_1 = 2u^2, t_2 = (\frac{8}{5})u\}.$$

The real sections of those sectors will be an interesting object to study.

Remark 1.2. A stratification of a certain (compact) manifold is an interesting problem of algebraic geometry, and has several intimate connections with the study of integrable systems. For example, the isospectral manifolds of the periodic Toda lattices are given by Jacobian varieties which have the stratification based on the theta divisor [1, 2, 15]. Also in the case of finite nonperiodic Toda lattices, the isospectral manifolds are compactified in the flag manifolds, and are decomposed into the Bruhat cells which correspond to the sets of singular solutions [6, 11]. (See also [22] for several examples of the stratifications of Jacobian varieties related to finite-dimensional Hamiltonian systems.) The present study will provide an example of the stratification related to integrable systems of hydrodynamic type.

In this letter, we construct an integrable hierarchy defined on each singular sector Z_m . These hierarchies can also be obtained by the dispersionless limits of the hidden hierarchies of the KdV equation considered in [19]. We call them the hidden BH hierarchies, and we construct them as a regularization of the shock singularity of the BH hierarchy. Each hierarchy on Z_m is called the BH_m hierarchy. In section 2, we discuss the Grassmannian structure of the singular sectors and show the connection to hyperelliptic curves. Then in section 3, we define the BH_m hierarchy on Z_m as a deformation of a hyperelliptic curve of genus m . We also show that the hierarchy can be put into the Riemann invariant form, and then construct the hodograph solution. The Riemann invariant form has also been obtained in a different approach in [9]. In section 4, we give a brief discussion on several extensions of the present construction of the hidden dispersionless hierarchies. We also mention some related topics such as a topological field theory [4] and a possible stratification of the Frobenius manifold as obtained by those hidden hierarchies [14].

2. Grassmannian structure of the BH hierarchy

In this section, we discuss the Grassmannian structure of the solutions of the BH hierarchy. Let us first recall (see for example [13]) that the BH hierarchy can be formulated as

$$\frac{\partial p}{\partial t_j} = \frac{\partial}{\partial x} Q_j \quad \text{with} \quad Q_j = (\lambda^{2j+1})_{+p} \quad j = 0, 1, 2, \dots \tag{2.1}$$

where $(\lambda^n)_{+p}$ indicates the polynomial part of λ^n in p having the algebraic relation

$$p^2 = \lambda^2 + u. \tag{2.2}$$

The hierarchy (2.1) can be considered as a deformation of the curve (2.2) which corresponds to the Riemann surface of genus 0 (sphere) with the compactification.

Later we will use $k = \lambda^2$, and then we assign the degrees as

$$\text{Deg}(k) = 2 \quad \text{Deg}(p) = 1.$$

One introduces a function $S(x, t_1, t_2, \dots)$ which plays an important role in the theory of dispersionless hierarchy. The S function is defined by rewriting (2.1) in the form

$$\frac{\partial S}{\partial t_j} = Q_j \quad \text{for} \quad j = 0, 1, 2, \dots \tag{2.3}$$

Since $Q_j = (\lambda^{2j+1})_{+p}$ is a polynomial of degree $2j + 1$ in p and $p^2 = k + u, k = \lambda^2$, the S function can be expressed by a Laurent series in $\mathbb{C}[[k, k^{-1}]] \cdot \sqrt{k}$,

$$S(t_0, t_1, t_2, \dots) = \sum_{i=0}^{\infty} k^{i+\frac{1}{2}} t_i + \sum_{j=0}^{\infty} \frac{1}{k^{j+\frac{1}{2}}} F_j(t_0, t_1, t_2, \dots).$$

To characterize the solution space of the BH hierarchy, we now define a linear space spanned by the flows (2.3) in a manner similar to the case of the KP hierarchy [3],

$$\begin{aligned} W_0 &:= \text{Span}_{\mathbb{C}} \{S_{t_0}, S_{t_1}, S_{t_2}, \dots\} \\ &= \text{Span}_{\mathbb{C}} \{Q_0, Q_1, Q_2, \dots\}. \end{aligned}$$

Since Q_i is a polynomial in p and $p^2 = k + u$, one can also consider Q_i to be an element in $\mathbb{C}[k] \cdot p$. This observation will be crucial for defining an integrable hierarchy on the singular sector. Thus we have

$$W_0 = \text{Span}_{\mathbb{C}} \{p, kp, k^2 p, \dots\} \cong \mathbb{C}[k] \cdot p.$$

This implies the inclusion

$$kW_0 \subset W_0$$

which is the condition for the KdV reduction in terms of the KP hierarchy [21].

One should however note that the linear space W_0 cannot be defined on the singular sectors where the flows (2.1) blow up and the commutativity among the S flows is not defined. Recall that the conditions on the hodograph solution $\Omega = 0$ provide a constraint on the coordinates. For example, the first condition $\Omega' = 0$ gives

$$c_1 t_1 + \sum_{j=2}^{\infty} j c_j u^{j-1} t_j = f'(u)$$

which implies that u can be determined as a function of t_j for $j \geq 1$. Then with $\Omega = 0$, the coordinate x can be considered as a function of the other times. Hence one can expect to have a map $\psi^m : \mathbb{C}^{\infty} \rightarrow Z_m$ so that

$$\psi^m(t_m, t_{m+1}, t_{m+2}, \dots) = (a_0, a_1, \dots, a_{m-1}, t_m, t_{m+1}, \dots)$$

where $a_i, 0 \leq i \leq m-1$ are some functions of $t_j, j \geq m$. The map can be defined at least locally on the sector Z_m , and it defines a patch on Z_m . The S -function then takes the form

$$S^m(t_m, t_{m+1}, \dots) = k^{m+\frac{1}{2}} \sum_{i=0}^{\infty} k^i t_{m+i} + \sum_{i=0}^{m-1} a_i(t_m, t_{m+1}, \dots) k^{i+\frac{1}{2}} + O\left(k^{-\frac{1}{2}}\right). \quad (2.4)$$

The linear space spanned by the flows $\partial S^m / \partial t_j, j \geq m$ under this restriction on the times defines

$$W_m = \text{Span}_{\mathbb{C}} \{S_{t_m}^m, S_{t_{m+1}}^m, S_{t_{m+2}}^m, \dots\} \cong k^m W_0.$$

Then introducing new variable $p_m := S_{t_m}^m$, we see W_m as the polynomial ring,

$$W_m \cong \mathbb{C}[k] \cdot p_m \quad \text{as a vector space.} \quad (2.5)$$

The Grassmannian Gr is a set of all linear spaces $W_m, m \geq 0$, and it has the Birkhoff decomposition [21]

$$Gr = \bigcup_{m \geq 0} \Sigma_m$$

where the Birkhoff stratum Σ_m is the set of all linear spaces of W_m . The structure of the Grassmannian is then the same as the KdV hierarchy by making the identification with $k = \lambda^2$,

$$W_m \cong \{\lambda^{-m}, \lambda^{-m+2}, \dots, \lambda^{m-2}, \lambda^m, \lambda^{m+1}, \dots\}$$

where $\lambda^2 W_m \subset W_m$ (i.e. $k W_m \subset W_m$) [3]. The BH hierarchy is thus defined on the principal stratum Σ_0 .

Since the degrees of k and p_m are assigned as

$$\text{Deg}(k) = 2 \quad \text{Deg}(p_m) = 2m + 1$$

one can consider a plane curve \mathcal{C}_m defined by

$$\mathcal{C}_m := \left\{ (k, p_m) \in \mathbb{C}^2 \mid p_m^2 = k^{2m+1} + \sum_{i=0}^{2m} u_i k^{2m-i} \right\} \tag{2.6}$$

which defines a hyperelliptic curve of genus m and is a smooth affine variety for the generic values of u_i , the deformation parameters. Then the ring $W_m = \mathbb{C}[k] \cdot p_m$ can be considered as a part of the quotient ring $\mathbb{C}[k, p]/\mathcal{C}_m$ with the curve \mathcal{C}_m . One should also note that there is a natural reduction of the curve,

$$\mathcal{C}_m \longrightarrow \mathcal{C}_{m-1} \quad (p_m = kp_{m-1})$$

which corresponds to the conditions on the parameters, $u_{2m} = u_{2m-1} = 0$. This can be considered as a dispersionless analogue of the regularization of the singularity in the KP hierarchy by the Bäcklund transformation [3]. Thus the stratum Σ_m is parametrized by the curve \mathcal{C}_m , and the boundary of the stratum corresponds to a singular curve of degenerate genus m . The higher order stratum is obtained by the desingularization of the curve by increasing the genus. The curve \mathcal{C}_m can be obtained by the dispersionless limit of the hidden KdV hierarchy considered in [19] and also by a quasi-classical $\bar{\partial}$ dressing method introduced in [18]. In the next section, we will consider a deformation of the curve by defining a system of equations for the parameters u_i in \mathcal{C}_m where the deformation is parametrized by the times t_j for $j \geq m$.

3. The hidden BH hierarchies

We now construct an integrable deformation of the curve \mathcal{C}_m in (2.6), which is defined on the singular sector Z_m . First recall the quotient ring of $\mathbb{C}[k, p_m]$ over an ideal \mathcal{C}_m which has a split,

$$\mathcal{R}_m := \frac{\mathbb{C}[k, p_m]}{\mathcal{C}_m} = \mathbb{C}[k] \oplus \mathbb{C}[k] \cdot p_m.$$

Then a deformation of the curve \mathcal{C}_m with an infinite number of deformation parameters is defined in the same form as (2.3),

$$\frac{\partial S^m}{\partial t_j} = Q_j^m \quad Q_j^m \in \mathbb{C}[k] \cdot p_m \quad j = 0, 1, 2, \dots \tag{3.1}$$

where S^m is given by (2.4), and $\text{Deg}(Q_j^m) = 2(m + j) + 1$ with $Q_0^m = p_m$. With this definition, the functions a_i in S^m can be explicitly written in terms of u_i in the curve \mathcal{C}_m . Here we have relabelled the times as $t_{m+i} \rightarrow t_i$, but we do not think it will cause any confusion. As we will show below, all the flows in (3.1) are compatible for appropriate form of Q_j^m , that is $\partial Q_i^m / \partial t_j = \partial Q_j^m / \partial t_i$, and in particular we have for the case $i = 0$

$$\frac{\partial p_m}{\partial t_j} = \frac{\partial}{\partial x} Q_j^m \tag{3.2}$$

where we denote $x = t_0$. This provides the system of equations of hydrodynamic type for u_i . Also note that the flows (3.2) are compatible with the automorphism of the curve, $p_m \rightarrow -p_m$.

Now we show the compatibility of the flows in (3.1). We first note

Lemma 3.1. *The Q_i^m in (3.1) are given by*

$$Q_i^m = \left(\frac{k^{m+i+\frac{1}{2}}}{p_m} \right)_+ p_m \quad \text{for } i = 0, 1, 2, \dots$$

where $(\cdot)_+$ denotes the projection onto $\mathbb{C}[k]$.

Proof. Since $\partial p_m / \partial t_i \sim O(k^{m-\frac{1}{2}})$, one can set

$$Q_i^m \sim k^{m+i+\frac{1}{2}} + O(k^{m-\frac{1}{2}}).$$

This implies

$$\frac{Q_i^m}{p_m} \sim \frac{k^{m+i+\frac{1}{2}}}{p_m} + O(k^{-1}).$$

The statement of the lemma then follows from $Q_i^m / p_m \in \mathbb{C}[k]$. □

This formula is a non-zero genus extension of the case with $m = 0$. In fact we have

Corollary 3.1.

$$Q_i = \left(k^{i+\frac{1}{2}} \right)_{+p} = \left(\frac{k^{i+\frac{1}{2}}}{p} \right)_+ p \quad \text{with } p^2 = k + u$$

where Q_i is defined in (2.1) with $k = \lambda^2$.

Proof. We first note

$$\left(\frac{k^{i+\frac{1}{2}}}{p} \right)_+ p = k^{i+\frac{1}{2}} - \left(\frac{k^{i+\frac{1}{2}}}{p} \right)_- p$$

where $(\cdot)_-$ is the non-polynomial part in k . Since $\text{Deg}(k) = 2$ and $\text{Deg}(p) = 1$, we have

$$\left(\frac{k^{i+\frac{1}{2}}}{p} \right)_- p = \left(\left(\frac{k^{i+\frac{1}{2}}}{p} \right)_- p \right)_{-p} = \left(k^{i+\frac{1}{2}} \right)_{-p}.$$

This implies the result. □

In order to show the compatibility of the flows (3.1), we note

Lemma 3.2. *The flows (3.1) can be put into the Lax form,*

$$\frac{\partial k}{\partial t_i} = \{Q_i^m, k\} := \frac{\partial Q_i^m}{\partial p_m} \frac{\partial k}{\partial x} - \frac{\partial Q_i^m}{\partial x} \frac{\partial k}{\partial p_m} \tag{3.3}$$

where p_m is then considered to be a constant parameter.

Proof. The flows $\partial p_m / \partial t_i = \partial Q_i^m / \partial x$ in (3.1) with $x = t_0$ can be written in terms of the differential three forms,

$$dp \wedge dk \wedge dx = dQ_i^m \wedge dt_i \wedge dk.$$

Then assuming $k = k(p, x, t)$, we obtain the result. □

It is standard that the following proposition proves the compatibility of the flows, that is $\partial^2 k / \partial t_i \partial t_j = \partial^2 k / \partial t_j \partial t_i$:

Proposition 3.1.

$$\frac{\partial Q_i^m}{\partial t_j} - \frac{\partial Q_j^m}{\partial t_i} + \{Q_i^m, Q_j^m\} = 0. \tag{3.4}$$

Proof. Note that writing $Q_i^m = \beta_i^+ p_m$ with $\beta_i^+ \in \mathbb{C}[k]$, equation (3.4) can be expressed by the following equation for β_i^+ ,

$$\frac{\partial \beta_i^+}{\partial t_j} - \frac{\partial \beta_j^+}{\partial t_i} + \{\{\beta_i^+, \beta_j^+\}\} = 0$$

where

$$\{\{f, g\}\} := f \frac{\partial g}{\partial x} - g \frac{\partial f}{\partial x} + p_m \{f, g\}.$$

This equation can be proved as follows: first note

$$\frac{\partial \beta_i}{\partial t_j} = \{\{\beta_j^+, \beta_i\}\} \quad \text{for } \beta_i = \frac{k^{m+i+\frac{1}{2}}}{p_m} \quad (\beta_j^+ = (\beta_j)_+).$$

With the decomposition $\beta_i = \beta_i^+ + \beta_i^-$, we have

$$\frac{\partial \beta_i^+}{\partial t_j} + \frac{\partial \beta_i^-}{\partial t_j} = \{\{\beta_j^+, \beta_i^+\}\} + \{\{\beta_j^+, \beta_i^-\}\}.$$

We also have

$$\frac{\partial \beta_j^+}{\partial t_i} + \frac{\partial \beta_j^-}{\partial t_i} = -\{\{\beta_i^-, \beta_j^+\}\} - \{\{\beta_i^-, \beta_j^-\}\}$$

where we have used $\{\{\beta_i, \beta_j\}\} = 0$. Combining these two equations and projecting on $\mathbb{C}[k]$, we obtain the compatibility equation for β_i^+ as well as β_i^- . This completes the proof. \square

The simplest example of the BH_m equations in terms of u_i is given by the t_1 -flow of the BH_1 equation (see remark 3.2 below for the general form),

$$\frac{\partial}{\partial t_1} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}u_0 & 1 & 0 \\ -u_1 & -\frac{1}{2}u_0 & 1 \\ -u_2 & 0 & -\frac{1}{2}u_0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}.$$

As a consequence of the Lax form (3.3), we have

Corollary 3.2. *The BH_m hierarchy (3.3) can be put into the Riemann invariant form,*

$$\frac{\partial \kappa_i}{\partial t_j} = \phi_i^j \frac{\partial \kappa_i}{\partial x} \quad \text{for } i = 0, 1, 2, \dots, 2m \tag{3.5}$$

where the Riemann invariants κ_i are the roots of the polynomial associated with the curve C_m , and ϕ_i^j is given by the $(\partial Q_j^m / \partial p_m)(k)$ at $k = \kappa_i$, that is

$$p_m^2 = \prod_{i=0}^{2m} (k - \kappa_i) \quad \text{and} \quad \phi_i^j = \phi^j(\kappa_i) = \left(\frac{k^{j+m+\frac{1}{2}}}{p} \right) \Big|_{k=\kappa_i}.$$

Proof. First note that from the curve $p_m^2 = F(k)$ we have

$$2p_m = \frac{\partial F}{\partial k} \cdot \frac{\partial k}{\partial p_m}$$

from which $\partial k/\partial p_m = 0$ implies $p_m = 0$. Then the evaluation of (3.3) at the roots of $p_m^2 = 0$ completes the proof. \square

For example, the Riemann invariant form for the first flow of the BH_m equation is given by

$$\frac{\partial \kappa_i}{\partial t_1} = \left(\kappa_i - \frac{1}{2}u_0 \right) \frac{\partial \kappa_i}{\partial x} \quad i = 0, 1, \dots, 2m$$

where $u_0 = -\sum_{i=0}^{2m} \kappa_i$. The Riemann invariant form has been obtained in [9]. We here note that the Riemann invariant form has an obvious reduction with $\kappa_j = 0$ for some j . This corresponds to the reduction of the genus as mentioned at the end of the previous section. There is also another reduction with all $\kappa_0 = \dots = \kappa_{2m}$ which is associated with a degenerate case of \mathcal{C}_m equivalent to $p_m^2 = k^{2m+1}$ of genus 0.

With form (3.5), we can construct the hodograph solution of the hierarchy as shown in [13]. We first remark that the functions $\phi^j(k)$ in (3.5) form a finite ring of dimension $2m + 1$,

$$\mathcal{F}_m := \frac{\mathbb{C}[k]}{p_m^2} = \text{Span}_{\mathbb{C}}\{\phi^0, \phi^1, \dots, \phi^{2m}\}. \tag{3.6}$$

Note here that $\phi^j(k)$ is a monic polynomial of $\text{Deg}(\phi^j) = 2j$ with $\text{Deg}(k) = 2$ and $\phi^0 = 1$. Then one can write the higher ϕ^{2m+j} for $j \geq 1$ in the form

$$\phi^{2m+j} = \sum_{i=0}^{2m} \mu_i^j(\kappa) \phi^i \pmod{p_m^2}$$

where $\mu_i^j(\kappa)$ are functions of the roots κ_i of $p_m^2 = 0$. We also note that the $\mu_i^j(\kappa)$ are functions of the symmetric polynomials v_i of the roots κ_j , which are given by $v_i = (-1)^i u_{i-1}$, $i = 1, \dots, 2m + 1$ with

$$v_1 = \sum_{j=0}^{2m} \kappa_j \quad v_2 = \sum_{i < j} \kappa_i \kappa_j \quad v_3 = \sum_{i < j < l} \kappa_i \kappa_j \kappa_l \dots v_{2m+1} = \prod_{i=0}^{2m} \kappa_i.$$

For example, we have for $j = 1$

$$\mu_{2m}^1 = -\frac{3}{2}u_0 \quad \mu_{2m-1}^1 = -\frac{3}{2}u_1 - \frac{3}{8}u_0^2 \quad \mu_{2m-2}^1 = -\frac{3}{2}u_2 - \frac{3}{4}u_0 u_1 + \frac{1}{16}u_0^3 \dots$$

Then we have

$$\frac{\partial \kappa_l}{\partial t_{2m+j}} = \sum_{i=0}^{2m} \mu_i^j(\kappa) \frac{\partial \kappa_l}{\partial t_i} \quad l = 0, 1, \dots, 2m$$

which implies that all the roots κ_l are constants along the characteristic curve,

$$\frac{dt_0}{\mu_0^j(\kappa)} = \dots = \frac{dt_{2m}}{\mu_{2m}^j(\kappa)} = \frac{dt_{2m+j}}{-1} \quad \text{for all } j \geq 1.$$

The integration of this equation gives

$$t_i^0 = t_i + \sum_{j=1}^{\infty} \mu_i^j(\kappa) t_{2m+j} \quad i = 0, 1, \dots, 2m \tag{3.7}$$

where t_i^0 is the initial position of the characteristic at $t_{2m+j} = 0$, $j \geq 1$. As a simplest example with $m = 1$, we have an explicit solution from (3.7) by setting $t_{2m+j} = (\frac{3}{2})\delta_{j1}$ and $t_i^0 = 0$,

$$u_0 = t_2 \quad u_1 = t_1 - \frac{1}{4}t_2^2 \quad u_2 = x - \frac{1}{2}t_1 t_2 + \frac{1}{6}t_2^3.$$

Also taking the sum $\sum_{i=0}^{2m} \phi^i t_i^0$ of (3.7), we obtain

Theorem 3.1. *The hodograph solution of the BH_m hierarchy is given by*

$$x + \sum_{i=0}^{\infty} \phi^i(\kappa_j) t_i = 0 \quad j = 0, 1, \dots, 2m \tag{3.8}$$

where the times (x, t_1, t_2, \dots) can be shifted arbitrary.

This formula will be useful in studying the singular structure of the BH_m hierarchy as in the case of the BH hierarchy. We will report a detailed analysis elsewhere.

Remark 3.2. The general form of the BH_m hierarchy can be written in the following form with the polynomial $\phi^i(k)$ for k being replaced by a matrix K ,

$$\frac{\partial U}{\partial t_i} = \phi^i(K) \frac{\partial U}{\partial x} \quad \text{with } U = (u_0, u_1, \dots, u_{2m})^T$$

where K is the (companion) matrix given by

$$K = \begin{pmatrix} -u_0 & 1 & \cdots & \cdots & 0 \\ -u_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -u_{2m-1} & 0 & \cdots & \cdots & 1 \\ -u_{2m} & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

For example, $\phi^1(K) = K - (\frac{1}{2})u_0 I$ with $I = (2m + 1) \times (2m + 1)$ identity matrix.

4. Further extensions and discussion

The present construction of the hidden hierarchies can be extended to the case with the hyperelliptic curve associated with an *even* degree polynomial,

$$\tilde{C}_m := \left\{ (k, p) \in \mathbb{C}^2 \mid p^2 = k^{2m+2} + \sum_{i=1}^{2m+2} v_i k^{2m+2-i} \right\}.$$

Here the degree of k is assigned as $\text{Deg}(k) = 1$ so that $\text{Deg}(p) = m + 1$. The hierarchy of deformations of the curve \tilde{C}_m can also be obtained as the dispersionless limit of the integrable hierarchy associated with the energy-dependent Schrödinger potentials considered in [20], which coincides with the hidden hierarchy for the Jaulent–Miodek hierarchy [12] (see also [17]). We here call the integrable hierarchy associated with the curve \tilde{C}_m the dJM_m hierarchy. Most of the results obtained in sections 2 and 3 remain the same for this case. In particular, the singular sectors Z_m for the dJM_0 hierarchy are given by the higher order intersections of two curves given by the hodograph solution for v_1 and v_2 , which are defined as follows: let the curve \tilde{C}_0 be given by

$$p^2 = k^2 + v_1 k + v_2 = (k - \kappa_1)(k - \kappa_2).$$

The hodograph solution has the same form as (1.1),

$$\Omega^j := x + \sum_{i=0}^{\infty} \phi^i(\kappa_j) t_i = 0 \quad j = 1, 2.$$

Then the singular sectors are defined by

$$Z_m = \left\{ (t_0, t_1, t_2, \dots) \in \mathbb{C}^\infty \mid (\kappa_1, \kappa_2) \in U_j, m = \sum_{j=1}^{\infty} j |U_j| \right\}$$

where U_j are defined by

$$U_j = \{(\kappa_1, \kappa_2) \mid \Omega^1 = \Omega^2 = 0, \tilde{\Omega}_{(i)} = 0 (1 \leq i \leq j), \tilde{\Omega}_{(j+1)} \neq 0\}.$$

Here the function $\tilde{\Omega}_{(i)}(\kappa_1, \kappa_2)$ may be given by $\tilde{\Omega}_{(i)} = D|_{\omega=\Omega_{(i)}^1/\Omega_{(i)}^2}$ with

$$D := \begin{vmatrix} (\Omega_1^1)^i & (\Omega_1^2)^i \\ \Omega_{(i)}^1 & \Omega_{(i)}^2 \end{vmatrix} \quad \Omega_{(i)}^j = \left(\frac{\partial}{\partial \kappa_2} - \omega(\kappa_1, \kappa_2) \frac{\partial}{\partial \kappa_1} \right)^i \Omega^j$$

where $\Omega_j^i = \partial \Omega^i / \partial \kappa_j$. Note that $\tilde{\Omega}_{(1)}$ is just the Jacobian determinant. The singular structure of the dispersionless JM hierarchy is then similar to the case of the BH hierarchy. We will report a detailed discussion on the stratification determined by the dJM_m hierarchies elsewhere.

One should also note that the integrable deformation of the algebraic curve defined by either the BH_m or dJM_m hierarchy is rather a general property of the dispersionless integrable hierarchy. For example, the dispersionless limit of the first hidden KP hierarchy discussed in [16] leads to the standard form of the elliptic curve,

$$C_{\text{KP}} : p^2 + v_0 p q + v_1 p = q^3 + u_0 q^2 + u_1 q + u_2$$

and the corresponding deformation has the form

$$\begin{cases} \frac{\partial p}{\partial t_n} = \frac{\partial}{\partial y} Q_n \\ \frac{\partial q}{\partial t_n} = \frac{\partial}{\partial x} Q_n \end{cases} \quad \text{with} \quad Q_n \in \frac{\mathbb{C}[p, q]}{C_{\text{KP}}} \quad n = 0, 1, 2, \dots$$

where $\text{Deg}(Q_n) = n + 2$ with $\text{Deg}(p) = 3$, $\text{Deg}(q) = 2$ and $t_0 = x$, $t_1 = y$. This contains the BH₁ hierarchy as the reduction with $q = k = \text{constant}$ with $v_0 = v_1 = 0$, and the hidden dispersionless Boussinesq hierarchy with the reduction $p = k = \text{constant}$ and $u_0 = 0$. The singular structures determined by the hidden dispersionless KP hierarchy and its reductions (such as the Gel'fand–Dikii type) will be discussed elsewhere. We also remark that the integrable deformation of hyperelliptic curve discussed in the present letter is different from that given by the Whitham hierarchy (see for example [10]). The Whitham hierarchy describes a slow modulation over a quasi-periodic solution of the original dispersive equation, and is obtained as a dispersive regularization of the dispersionless hierarchy. However, our deformation describes each stratum (singular sector) of the Birkhoff decomposition, which is parametrized by the hyperelliptic curve, and it has no direct connection with the original dispersive equation.

As a final remark, we would like to point out that the BH_m hierarchy may not have a single free energy (or pre-potential) which plays a central role in a topological field theory. In the dispersionless KP hierarchy, the free energy can be obtained by integrating twice the function G_{ij} (referred to as the Gel'fand–Dikii potential in [4]) defined as the flux density of the conservation laws,

$$\frac{\partial g_j}{\partial t_i} = \frac{\partial}{\partial x} G_{ij}. \quad (4.1)$$

Here $g_j = G_{0j}$ and G_{ij} are the coefficients of the expansions of p_m and Q_i^m in (3.2). In the case of the BH_m, they are given by the residue formulae,

$$G_{ij} = \text{Res}_{k=\infty} \left(Q_i^m k^{j-m} \frac{dk}{\sqrt{k}} \right).$$

Because of the conservation law (4.1), G_{ij} can be integrated once to get the form with some function F_j ,

$$G_{ij} = \frac{\partial}{\partial t_i} F_j.$$

(Note that F_j are related to the functions a_i in (2.4) as $F_j = a_{m-1-j}$ for $0 \leq j \leq m-1$.) However, one can show that G_{ij} is not a symmetric function with respect to the indices i, j except the case with $m = 0$. Hence there is no further integration to express the G_{ij} as a two-point function as in the case of dispersionless KP theory (see for example [4, 14]), such as

$$G_{ij} = \langle \phi^i \phi^j \rangle = \frac{\partial^2}{\partial t_i \partial t_j} F$$

where ϕ_i are the primary fields or gravitational descendants and F represents the free energy of the topological field theory. Thus the BH_m hierarchy is somewhat different from most of the known dispersionless hierarchies which have a direct connection to topological field theory [4, 7]. It is then quite interesting to construct a topological field theory based on the BH_m hierarchies, and if there exists such a theory, then the associated Frobenius manifold may have a stratified structure characterized by a finite ring related to \mathcal{F}_m in (3.6).

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